The Spin-Flavor Dependence of Nuclear Forces From Large-N QCD

David B. Kaplan

Institute for Nuclear Theory, University of Washington

Box 351550, Seattle WA 98195-1550, USA

`dbkaplan@phys.washington.edu`

and

Martin J. Savage [†]

Department of Physics, Carnegie Mellon University

Pittsburgh PA 15213, USA

`savage@thepub.phys.cmu.edu`

We show that nuclear interactions are $SU(4)$ symmetric at leading order in chiral perturbation theory in the large-N limit of QCD. The nucleons and delta resonances form a 20-dimensional representation of $SU(4)$ and we show how Wigner's supermultiplet symmetry $SU(4)_{sm}$, under which the nucleons transform as a 4-dimensional representation, follows as an accidental low energy symmetry. Exploiting $SU(4)$ symmetry allows one to express the 18 independent leading N, Δ interaction operators invariant under $SU(2)_I \times SU(2)_J$ in terms of only two couplings. The three flavor analogue allows one to express the 28 leading octet, decuplet interactions in terms of only two couplings, which has implications for hypernuclei and strangeness in "neutron" stars.

9/95

[†] DOE Outstanding Junior Investigator

1. Implications of spin-flavor symmetry in effective nuclear forces

Short distance nuclear forces relevant for low energy processes can be incorporated into chiral Lagrangians in terms of local operators in a derivative expansion [1,2]. There are two leading (dimension six) operators involving nucleons alone, given by

$$\mathcal{L}_6 = -\tfrac{1}{2}C_S(N^\dagger N)^2 - \tfrac{1}{2}C_T(N^\dagger \vec{\sigma} N)^2 \qquad (1.1)$$

where N are isodoublet two-component spinors, and the $\vec{\sigma}$ are Pauli matrices. Higher derivative operators account for the spin-orbit coupling, among other effects[1]. Including the Δ isobars in the theory leads to 18 independent dimension six operators allowed by spin and isospin symmetry[2]. In order to discuss hypernuclei, or strangeness in dense matter, one must consider $SU(3)$ flavor multiplets — there are six independent leading operators involving the baryon octet alone [3], while including the decuplet inflates the number to 28 independent operators. The number of independent dimension seven interactions is still much greater.

Clearly, to make headway in a systematic effective field theory analysis of nuclear and hypernuclear forces, it is desirable to find some simplifying principle. In this letter we propose that among the baryon interactions, $SU(4)$ spin-flavor symmetry for two flavors, or $SU(6)$ symmetry for three flavors should be a good approximation. We show how these symmetries have a vastly simplifying effect on the dimension six interactions described above, reducing both the 18 $N-\Delta$ interactions and the 28 octet-decuplet interactions down to just two independent operators. We support our allegation that spin-flavor symmetry is relevant to nuclear forces first by outlining its implications and by giving empirical evidence in support of $SU(4)$ in nuclei. Then we prove that these symmetries become exact in the large-N limit of QCD.

---

[1] In low energy nucleon-nucleon scattering the higher derivative terms will be less important than the leading operator. However, many-body effects in large nuclei can enhance the importance of subleading operators, such as the spin-orbit interaction.

[2] It is simplest to count operators in the form $(\psi_1\psi_2)(\psi_3\psi_4)^\dagger$, requiring $(\psi_1\psi_2)$ and $(\psi_3\psi_4)$ to have the same spin and isospin quantum numbers. One finds the above two $(NN)(NN)^\dagger$ operators; zero operators of the form $(NN)(N\Delta)^\dagger$; two $(NN)(\Delta\Delta)^\dagger$, four $(N\Delta)(N\Delta)^\dagger$, two $(N\Delta)(\Delta\Delta)^\dagger$, and eight $(\Delta\Delta)(\Delta\Delta)^\dagger$ operators.

Under $SU(2f)$ symmetry the two spin states and f flavors of quarks transforming as the $2f$ dimensional defining representation. For $N = 3$, the lowest lying baryons have the quantum numbers of three quarks in an S-wave, transforming as a three index symmetric tensor $\Psi^{\mu\nu\rho}$ under $SU(2f)$. For $f = 2$, Ψ is the 20-dimensional representation of $SU(4)$, comprising of the four N and sixteen Δ spin/isospin states; for $f = 3$, Ψ is the 56-dimensional $SU(6)$ representation containing the $J = \frac{1}{2}$ octet and the $J = \frac{3}{2}$ decuplet. In either case one finds that there are only two $SU(2f)$ invariant dimension six operators. These can be written in terms of the baryon fields Ψ as

$$\mathcal{L}_6 = -\frac{1}{f_\pi^2} \left[a(\Psi^\dagger_{\mu\nu\rho}\Psi^{\mu\nu\rho})^2 + b\Psi^\dagger_{\mu\nu\sigma}\Psi^{\mu\nu\tau}\Psi^\dagger_{\rho\delta\tau}\Psi^{\rho\delta\sigma} \right] \ , \tag{1.2}$$

where $f_\pi = 132$ MeV is the pion decay constant.

Eq. (1.2) can be expressed in terms of the more familiar fields by writing each $SU(2f)$ index μ as a pair of flavor and spin indices $(i\alpha)$ under $SU(f) \times SU(2)_J$, and then by projecting out components with the desired $SU(f) \times SU(2)_J$ transformation properties. In this way one finds for two flavors

$$\Psi^{(\alpha i)(\beta j)(\gamma k)} = \Delta^{ijk}_{\alpha\beta\gamma} + \frac{1}{\sqrt{18}} \left(N^i_\alpha \epsilon^{jk}\epsilon_{\beta\gamma} + N^j_\beta \epsilon^{ik}\epsilon_{\alpha\gamma} + N^k_\gamma \epsilon^{ij}\epsilon_{\alpha\beta} \right) \ , \tag{1.3}$$

and for three flavors

$$\Psi^{(\alpha i)(\beta j)(\gamma k)} = T^{ijk}_{\alpha\beta\gamma} + \frac{1}{\sqrt{18}} \left(B^i_{m,\alpha} \epsilon^{mjk}\epsilon_{\beta\gamma} + B^j_{m,\beta} \epsilon^{mki}\epsilon_{\gamma\alpha} + B^k_{m,\gamma} \epsilon^{mij}\epsilon_{\alpha\beta} \right) . \tag{1.4}$$

In the above expressions N, Δ, B, and T are the nucleon, isobar, octet and decuplet fields respectively. Indices $i, j \ldots$ denote flavor indices while $\alpha, \beta \ldots$ are spin indices. Δ and T are totally symmetric tensors separately in flavor and spin; B is a traceless matrix in flavor. The normalization factors of $1/\sqrt{18}$ are fixed so that the baryon number operator equals $(\Psi^\dagger_{\mu\nu\rho}\Psi^{\mu\nu\rho})$.

By plugging expressions (1.3), (1.4) into our $SU(2f)$ symmetric Lagrangian (1.2) it is possible to determine all of the leading short range interactions between the $J = \frac{1}{2}$ and $J = \frac{3}{2}$ baryons in terms of the two coefficients a and b. We spare the reader all of the results, and focus on predictions for interactions solely involving the $J = \frac{1}{2}$ baryons. For

two flavors, the $SU(4)$ symmetry yields predictions for the Weinberg coefficients of eq. (1.1) in terms of a and b:

$$C_S = \frac{2(a - b/27)}{f_\pi^2} \ , \qquad C_T = 0 \qquad (two\ flavors). \qquad (1.5)$$

For three flavors, $SU(6)$ symmetry predicts the six Savage-Wise (SW) coefficients $c_1 \ldots c_6$ [3] for the interactions involving four baryon octet fields:

$$\begin{array}{ll}
c_1 = -\frac{7}{27}b & c_4 = -\frac{14}{81}b \\[4pt]
c_2 = \frac{1}{9}b & c_5 = a + \frac{2}{9}b \qquad (three\ flavors)\ . \qquad (1.6) \\[4pt]
c_3 = \frac{10}{81}b & c_6 = -\frac{1}{9}b
\end{array}$$

Given that $C_S = (2c_1 + c_2 + 2c_5 + c_6)/f_\pi^2$ and $C_T = (c_2 + c_6)/f_\pi^2$, the $SU(6)$ prediction (1.6) contains the $SU(4)$ prediction (1.5). Note that the contributions proportional to b are quite suppressed. In fact, as we discuss below, in the large-N limit of QCD, both a/f_π^2 and b/f_π^2 are $\mathcal{O}(N)$, while there are $\mathcal{O}(1)$ ($\sim 30\%$) violations to the $SU(6)$ predictions. Similarly one expects $\sim 30\%$ violations due to $SU(3)$ breaking by the strange quark mass. Thus the contributions proportional to b in the above relations are subleading, and it is consistent to take $b = 0$ in eq. (1.6) to leading order in N. Note that with the b terms negligible, the self-interactions among the $J = \frac{1}{2}$ octet baryons are invariant under an accidental $SU(16)$ symmetry. For two flavors, the prediction $C_T = 0$ implies an accidental $SU(4)$ symmetry among the nucleon self-interactions.

2. Evidence for $SU(4)$ symmetry in nuclear physics

Why should one believe that there is an approximate $SU(2f)$ symmetry in the short-range nuclear forces? We first give empirical evidence for $SU(4)$ symmetry in nuclear physics; we then show that in the large-N limit of QCD, $SU(4)$ relations are accurate to order $1/N^2$ ($\sim 10\%$), while $SU(6)$ relations are accurate to $\mathcal{O}(1/N)$ ($\sim 30\%$) and $\mathcal{O}(m_s)$.

The testable consequence of $SU(4)$ symmetry is the prediction (1.5) that $C_T = 0$, which implies the existence of an *accidental* symmetry in low energy nucleon interactions. This symmetry is Wigner's "supermultiplet symmetry" [4], which we will denote $SU(4)_{sm}$;

3

it arises because with $C_T = 0$ the interaction (1.1) is simply proportional to $(N^\dagger N)^2$ which is invariant under an $SU(4)$ with the four spin and isospin nucleon states transforming as a 4-dimensional representation. Wigner's $SU(4)_{sm}$ is not equivalent to the more fundamental $SU(4)$ symmetry, under which the nucleon transforms as part of a 20-dimensional representation along with the deltas. Since $SU(4)_{sm}$ is broken by dimension seven operators (*e.g.* spin-orbit interactions) as well as dimension six $N - \Delta$ interactions, it is only expected to be an approximate symmetry in light nuclei.

The validity of $SU(4)_{sm}$ in light nuclei is supported by an array of evidence. An extensive discussion is given in ref. [5] ; additional support can be found in the literature in the context of electron scattering [6], giant dipole resonance multiplets [7], β-decay selection rules in $A = 19$ nuclei [8], and double β-decay [9]. A particularly illuminating discussion of $SU(4)_{sm}$ symmetry in nuclei is found in [10]. We will mention here two particular pieces of evidence for $SU(4)_{sm}$: the two nucleon system, and β-decays of $A = 18$ nuclei.

The two-nucleon system provides a striking example of $SU(4)_{sm}$ symmetry in nuclear interactions. The $I = 0$, $S = 1$ channel has a scattering length of $a_1 = 5.423$fm and a bound state (the deuteron) with binding energy of $E_B = 2.225$MeV. In contrast the $I = 1$, $S = 0$ channel has scattering lengths $a_0 \sim -20$fm, and the threshold states are nearly bound. In the $SU(4)_{sm}$ limit both channels would be identical, and so the very different scattering lengths would make it appear that $SU(4)_{sm}$ is badly broken. However, scattering lengths are very sensitive to small changes in the interaction potential if there are almost bound or almost unbound eigenstates, as is the case in the nucleon-nucleon system. Thus there may be an approximate symmetry in the interaction potentials which is not evident in the scattering lengths. Modelling the nuclear two-body potential by a spherically symmetric, finite depth square well one finds that the relevant potential parameters (V_0 the depth and R the range) are $V_{0t} = 38.5$MeV, $R_t = 1.93$fm for the spin-triplet state and $V_{0s} = 20.3$MeV, $R_s = 2.50$fm for the spin singlet states [11]. As we will show in the next section, the large-N analysis predicts that the interaction should be $SU(4)$ symmetric in the Born approximation; beyond Born approximation, infrared divergences can mask the result. If we match the Weinberg coefficients $C_{S,T}$ at scattering momenta considerably

above threshold (but below the scale where the derivative expansion breaks down) then the Born expansion in the full and effective theories should match pretty well. Thus $C_{S,T}$ are proportional to the spatial integrals of the nucleon-nucleon potential; in the case of the model of a spherically symmetric, square well potential described above, this leads to

$$
\begin{aligned}
C_S &= -\frac{\pi}{3}\left(3V_{0t}R_t^3 + V_{0s}R_s^3\right) &= -2.24\frac{1}{f_\pi^2} \\
C_T &= -\frac{\pi}{3}\left(V_{0t}R_t^3 - V_{0s}R_s^3\right) &= 0.126\frac{1}{f_\pi^2}
\end{aligned}
\tag{2.1}
$$

Evidently the nuclear two-body interaction is $SU(4)_{sm}$ invariant with corrections at the $C_T/C_S \simeq 5\%$ level, consistent with what one might expect from a $1/N^2$ effect with $N = 3$. Note that this fit fixes one linear combination of the a and b coefficients, as given in eq. (1.5). Weinberg chose to fit C_S and C_T at threshold, so that they were proportional to scattering lengths; this procedure yields $C_T/C_S \sim -4$, completely obscuring the approximate $SU(4)$ symmetry [1].

A second demonstration of $SU(4)_{sm}$ symmetry in nuclear interactions can be found in the strengths of Gamow-Teller transitions between light nuclei. The Gamow-Teller operator is an $SU(4)_{sm}$ generator and as such cannot induce transitions between states belonging to different irreducible representations of $SU(4)$. We have found a particularly compelling demonstration in transitions between the $A = 18$ nuclei, ^{18}O , ^{18}F and ^{18}Ne. The energy level diagram for the $A = 18$ nuclei and the observed β-decays are shown in Fig. 1, and can be found in more detail in [12]. The naive structure of these nuclei is two valence nucleons in the $s - d$ shell on a closed, inert ^{16}O core (the predominant mixing with deformed states is expected to involve four nucleons in an $SU(4)_{sm}$ singlet state in the $s - d$ shell and two holes in the core, which will not affect our classification of these states under $SU(4)_{sm}$). The results of shell-model computations [13,14], show that the two valence nucleons in the lowest lying $(J^\pi, I) = (0^+, 1)$ and $(1^+, 0)$ states are predominantly in a relative $L = 0$ configuration. This allows us to identify the ground states of ^{18}O , ^{18}F and ^{18}Ne and the excited state of ^{18}F at $E = 1.04$MeV (as shown in Fig. 1) as the members of the **6** representation of $SU(4)_{sm}$. (The ^{18}F ground state has $(J, I) = (1, 0)$, while the other three states form a $(J, I) = (0, 1)$ multiplet). In fact, shell-model computations indicate that these states are in the same $SU(4)_{sm}$ multiplet with
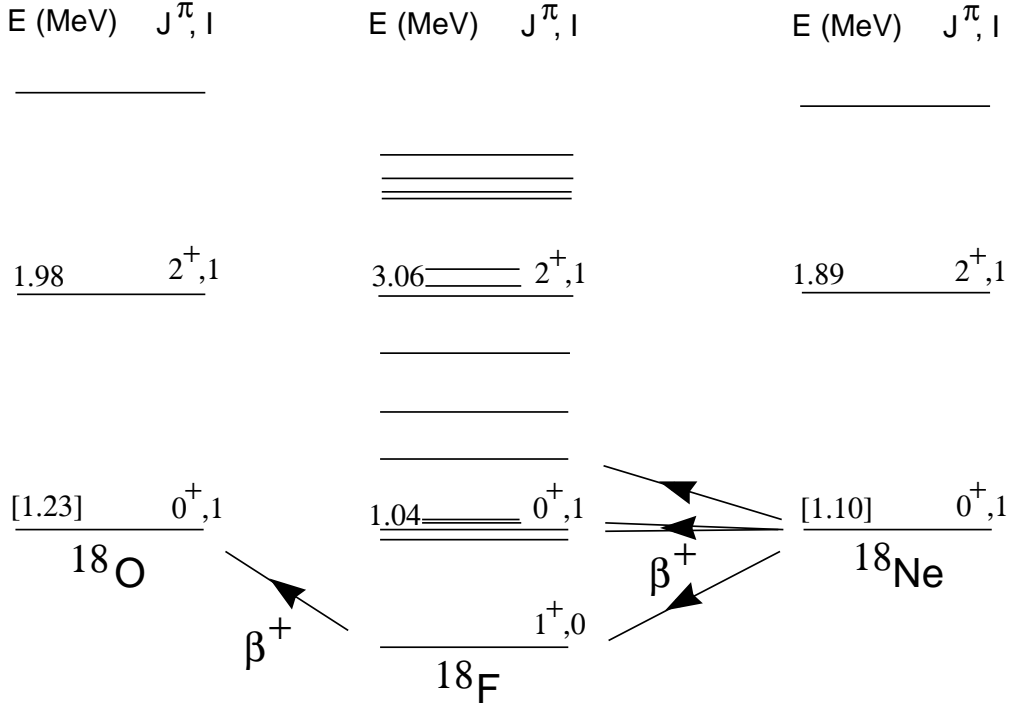
Fig. 1. *Energy level diagram for ^{18}O , ^{18}F and ^{18}Ne. The $n-p$ mass difference and coulomb effects have been removed. The energies in square brackets are relative to the ^{18}F ground state. β-decays to various states are indicated by arrows.*

a probability of 87% [15]; the "missing" 13% is largely due to the spin-orbit interaction. The β-decay strength of transitions between states of this supermultiplet along with the strengths of transitions to states outside the supermultiplet are shown in Table 1 [16].

It is seen from the first three entries of Table 1 that the Gamow-Teller transitions ($\propto \vec{\sigma}\tau_+$) to states within the $SU(4)$ supermultiplet are of similar strength as the superallowed transition ($\propto \tau_+$), $^{18}Ne \xrightarrow{\beta^+} {}^{18}F(1.04\text{MeV})$, despite the fact that they are not related by isospin. Further, it is clear from the last two entries that the decays to states outside the supermultiplet are much weaker than those to states within the supermultiplet. To be quantitative, we give the ratio of matrix elements (corrected for Coulomb interactions, g_A, and final state multiplicities) relative to the superallowed transition as the parameter R in the last column. Isospin predicts equality between the first two entries; $SU(4)_{sm}$ predicts equality between the first three entries. Allowing for a 13% effect due to contamination of

Decay	$(J^\pi, I) \rightarrow (J^\pi, I)$	$\log_{10}(ft)$	R
$^{18}F \xrightarrow{\beta^+} {}^{18}O(g.s)$	$(1^+, 0) \rightarrow (0^+, 1)$	3.554	0.73
$^{18}Ne \xrightarrow{\beta^+} {}^{18}F(g.s)$	$(0^+, 1) \rightarrow (1^+, 0)$	3.096 ± 0.004	0.71
$^{18}Ne \xrightarrow{\beta^+} {}^{18}F(1.04\text{MeV})$	$(0^+, 1) \rightarrow (0^+, 1)$	3.473 ± 0.013	1
$^{18}Ne \xrightarrow{\beta^+} {}^{18}F(1.08\text{MeV})$	$(0^+, 1) \rightarrow (0^-, 0)$	7.012 ± 0.059	0.017
$^{18}Ne \xrightarrow{\beta^+} {}^{18}F(1.70\text{MeV})$	$(0^+, 1) \rightarrow (1^+, 0)$	4.477 ± 0.015	0.15

Table 1: $\log_{10}(ft)$ *values for β-decay between A=18 nuclei. The first three decays are between states in a* **6** *of $SU(4)_{sm}$; the last two are decays between different $SU(4)_{sm}$ multiplets. R measures the matrix element relative to that for the third entry (corrected for phase space, final state multiplicity, g_A factors).*

the wavefunctions largely due to the spin-orbit interaction, we see that $SU(4)_{sm}$ is a good symmetry in the dimension six interactions to within $\sim 20\%$, again roughly consistent with $SU(4)_{sm}$ violation at the $1/N^2$ level [3].

3. Spin-flavor symmetry from large-N QCD

We now turn to a theoretical justification for spin-flavor symmetry in the leading nuclear forces, appealing to the $1/N$ expansion of QCD. The internal structure of baryons becomes greatly simplified as that the number of colors N becomes large. In this limit, baryons are comprised of N quarks and a baryon mass scales like N. As shown by 't Hooft, virtual $q\bar{q}$ pairs are suppressed in this limit [18]; subsequently Witten showed that the quarks in a baryon obey a relativistic Hartree equation [19]. For two flavors one can also prove certain $SU(4)$ relations among operator matrix elements in single large-N baryons states [20-24]. The existence of these relations is due to the fact that while large-N analogues of the nucleon have N quarks, they still have $\mathcal{O}(1)$ spin and isospin, so that the spin-dependent Hartree potential experienced by quarks is down by $1/N$ relative to

---

[3] If the contamination is greater than 13% — as is suggested by the fact that the shell model systematically overestimates Gamow-Teller strengths [17] — then the agreement with with the $SU(4)_{sm}$ prediction may be better than 20%.
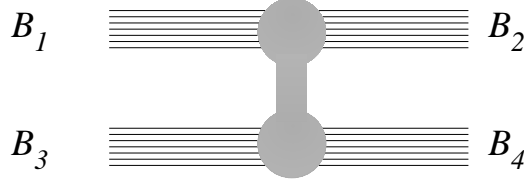
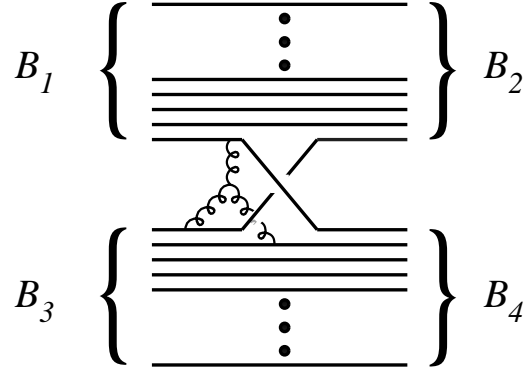Fig. 2. *A connected baryon-baryon interaction in large-N QCD due to the exchange of quarks and gluons.*

Fig. 3. *A connected diagram that is $\mathcal{O}(1/N^2)$ involving a 1-body operator between $\langle B_1 |$ and $|B_2 \rangle$, and a 2-body operator between $\langle B_3 |$ and $|B_4 \rangle$.*

the spin independent potential. With three flavors one finds $SU(6)$ relations, provided one considers baryons with $\mathcal{O}(1)$ strange quarks, as well as $\mathcal{O}(1)$ spin. $SU(4)$ relations among 1-baryon matrix elements are found to work to $\mathcal{O}(1/N^2)$ [21], while $SU(6)$ relations get $1/N$ corrections [21,24]; some of $1/N$ corrections can be computed, however [21,22]. We will find similar symmetries and corrections in the 2-baryon sector.

Witten discussed how best to consider baryon-baryon collisions [19]: one works with the time dependent Schrodinger equation in (relativistic) Hartree approximation. It is not possible to construct the Hartree potential explicitly, but we can deduce certain of its properties by making the reasonable assumption it is given by the sum of connected Feynman diagrams in a $1/N$ expansion. Such diagrams have the generic form of Fig. 2, where the blob represents the exchange of an arbitrary number of quarks between the two baryons, as well as gluon exchange between the quarks.

A specific example of such a diagram is given in Fig. 3; it is seen to be $\mathcal{O}(1/N^2)$ since each gluon coupling brings a factor of $1/\sqrt{N}$, and there are no closed color loops in the diagram. Since there are three quarks involved and N possible choices for each quark, the matrix element of this operator will involve a combinatoric factor of N^3, making its net contribution to the baryon-baryon scattering amplitude $\mathcal{O}(N)$. One can easily generalize to interactions involving any number of quark exchanges: leading connected graphs involving r quarks scale as $N^{(1-r)}$, and their contribution to the amplitude is $\mathcal{O}(N)$ [19].

Our analysis of the baryon-baryon scattering diagrams will use techniques and notation similar to those in [21-24] — most closely those of ref. [22]. It is convenient to classify the connected diagrams contributing to the interaction in Fig. 2 by the number of quarks n involved on the $B_1 - B_2$ baryon line, by the number of quarks n' involved on the $B_3 - B_4$ line, and by the total isospin and spin (I, J) transmitted between the two baryon lines [4]. In order to match onto the effective nonrelativistic operator (1.2) we work to zeroth order in the baryon velocities; thus the only source of angular momentum in the problem are the baryon spins. The spin-flavor dependence of a given diagram leading in $1/N$ is

$$N \, \langle B_2| \, \frac{\mathcal{O}_{IJ}^{(n)}}{N^n} \, |B_1\rangle \, \langle B_4| \, \frac{\overline{\mathcal{O}}_{IJ}^{(n')}}{N^{n'}} \, |B_3\rangle \tag{3.1}$$

where the operator $\mathcal{O}_{IJ}^{(n)}$ is an n-quark operator on the upper baryon line with isospin and spin (IJ), and $\overline{\mathcal{O}}^{(n')}$ is an n'-quark operator on the lower baryon line with the same (IJ) and conjugate I_3, J_3; the two operators are contracted so that the amplitude is a spin-isospin singlet. Note that the spin of a quark is defined in the collision center-of-momentum frame.

The first step in proving that the amplitudes in (3.1) imply $SU(4)$ symmetry is to prove that

$$\langle B'| \, \frac{\mathcal{O}_{IJ}^{(n)}}{N^n} \, |B\rangle \, \lesssim \, \frac{1}{N^{|I-J|}} \, , \tag{3.2}$$

---

[4] The identity of the two baryon lines can be kept distinct by considering the number of quarks involved n and n' to be $\mathcal{O}(1)$; diagrams with n or n' of $\mathcal{O}(N)$ are actually suppressed since the combinatoric factor for choosing n quarks is not N^n, but the much smaller binomial coefficient $\binom{N}{n}$.

for states B, B' that have $I \sim J \sim \mathcal{O}(1)$, and that operators that saturate the bound (3.2) have a particular form. Eq. (3.2) contains as a specific case the $I_t = J_t$ rule discussed in the Skyrme model [25]. The dominance of $I = J$ couplings is well known for the pion; it also is observed for the rho meson [26]. Eq. (3.2) was previously derived in ref. [22].

Matrix elements such as in eq. (3.1) can be represented by strings of one-quark operators acting on the baryon. Each of these 1-quark operators can have any of the 16 $SU(2)_I \times SU(2)_J$ quantum numbers appropriate for a $q\bar{q}$ pair: $(0,0)$, $(1,0)$, $(0,1)$ or $(1,1)$, and can be represented as

$$\mathbf{1} \, , \quad \mathbf{I}_a \, , \quad \mathbf{J}_i \, , \quad \mathbf{G}_{ia} \, . \tag{3.3}$$

The first three of these operators are simply the generators of quark number, isospin, and spin respectively. Taken together, the 16 operators (3.3) generate $SU(4) \times U(1)$ symmetry, with commutators of the generic form

$$[\mathbf{I}, \mathbf{I}] \sim \mathbf{I} \, , \quad [\mathbf{J}, \mathbf{J}] \sim \mathbf{J} \, , \quad [\mathbf{I}, \mathbf{G}] \sim [\mathbf{J}, \mathbf{G}] \sim \mathbf{G} \, , \quad [\mathbf{G}, \mathbf{G}] \sim \mathbf{I} + \mathbf{J} \, , \tag{3.4}$$

all other commutators vanishing. Each of these operators when acting on a baryon state $|B\rangle$ produces another baryon state $|B'\rangle$, since they do not change quark number. When $|B\rangle$ is a state with $I \sim J \sim \mathcal{O}(1)$, then the operators $\mathbf{I}$ and $\mathbf{J}$ produce $|B'\rangle$ with amplitude $\mathcal{O}(1)$, while the operators $\mathbf{1}$ and $\mathbf{G}$ produce $|B'\rangle$ with amplitude $\mathcal{O}(N)$. Thus the leading operators $\mathcal{O}^{(n)}$ are of the form

$$\mathcal{O}^{(n)} = \mathbf{G}^r \mathbf{1}^{n-r} \, . \tag{3.5}$$

Furthermore, while $\mathbf{G}^2$ generically has a matrix element that is $\mathcal{O}(N^2)$, from the commutation relations (3.4), the matrix element of $[\mathbf{G}_{ia}, \mathbf{G}_{jb}]$ has a matrix element $\mathcal{O}(1)$. Thus the $\mathbf{G}$'s in the leading operators (3.5) are totally symmetrized. Now consider how the indices of the $\mathbf{G}$ operators might be contracted. Using Fierz-type identities for Pauli matrices, one can show that [22]

$$\mathbf{G}_{ia}\mathbf{G}_{ib} = \delta_{ab} \left(\mathbf{1}\mathbf{1} - \mathbf{I}_c\mathbf{I}_c \right) + \mathbf{I}_a\mathbf{I}_b + (1 - \text{body operators}) \tag{3.6}$$

$$\mathbf{G}_{ia}\mathbf{G}_{jb}\epsilon_{abc}\epsilon_{ijk} = -2 \left(\mathbf{G}_{kc}\mathbf{1} - \mathbf{J}_k\mathbf{I}_c \right) + (1 - \text{body operators}) \tag{3.7}$$

10

and so the leading operators can always be written in the form (3.5) with the r $\mathbf{G}$ operators totally symmetrized, and with none of the $\mathbf{G}$ indices contracted. Such an operator has $I = J = r$. Similar arguments show that the largest n-quark operator with $I = J + t$ has the form $\mathbf{G}^J \mathbf{I}^t \mathbf{1}^{(n-J-t)}$ and has a matrix element of order $N^{(n-t)}$. This proves the assertion (3.2).

The same identity (3.7) which allowed us to ignore as subleading any antisymmetrized pair of spin or isospin indices allows us to independently rearrange those indices on either of the baryon lines. Thus we can write the leading large-N contributions to the baryon-baryon interaction (3.1) in the form

$$N \left(\frac{1}{N^2} \sum_{i,a=1}^{3} \{\mathbf{G}_{ia}\}_1 \{\mathbf{G}_{ia}\}_2 + \mathcal{O}(1/N^2) \right)^I \tag{3.8}$$

times any number of identity operators, where I equals the t-channel spin and isospin; the parentheses $\{\ \}_1$ and $\{\ \}_2$ refer to which of the two baryon lines the 1-quark operators act upon. (We omit the unit operators which do not modify the $SU(2)_J \otimes SU(2)_I$ structure). The proof of $SU(4)$ invariance now follows trivially. If we denote the fifteen $SU(4)$ generators by $T_\mu = \{\mathbf{I}_a, \mathbf{J}_i, \mathbf{G}_{ia}\}$ and use the fact that the matrix elements of I_a and J_i are $\mathcal{O}(1)$, it follows that the leading interaction (3.8) may be rewritten as

$$N \left(\frac{1}{N^2} \sum_{\mu=1}^{15} \{\mathbf{T}_\mu\}_1 \{\mathbf{T}_\mu\}_2 + \mathcal{O}(1/N^2) \right)^I \tag{3.9}$$

which is manifestly $SU(4)$ invariant, up to corrections suppressed by $1/N^2$. This shows that each individual connected graph contributing in Fig. 2, summed over colors, is $SU(4)$ symmetric.

Similar arguments go through for three flavors, and one finds that in the large-N limit of QCD with $SU(3)$ flavor symmetry, the low energy effective theory will have $SU(6)$ invariant baryon interactions (for low I, J, S states). For equal mass quarks, corrections to $SU(6)$ are of order $1/N$, however, rather than $1/N^2$ as in the two flavor case. One source of the $1/N$ correction is that diagrams involving transfer of a strange quark between baryons are subleading by $1/N$; another source is that the hypercharge matrix T_8 is proportional

to the unit operator plus $1/N$ corrections. In addition to the $1/N$ corrections, there will be $SU(6)$ violation proportional to powers of m_s, the strange quark mass. These corrections are potentially quite important for phenomenology.

So far we have not discussed the space dependence of the contributions from Fig. 2. We assume that the only long-range contribution comes from pion exchange, and that the remaining contributions can be Taylor expanded in terms of q^2/Λ^2, (Λ given roughly by the vector meson masses) in the spirit of chiral perturbation theory. We stress that the $SU(2f)$ invariance we found in the large-N limit holds separately for each connected contribution (3.1). This may seem counter-intuitive: $SU(4)$ symmetry in the quark model unites the π and the ρ mesons into a single multiplet. Since the ρ and π have such different masses, $SU(4)$ is badly broken in the meson sector. One might conclude that since meson exchange plays a big role in nuclear interactions, those interactions would have large $SU(4)$ violation as well. In fact, as far as the $SU(4)$ symmetry in nuclear forces goes, the π and ρ are not united into the same multiplet. Rather each couples to the baryons with strength N and coupling G_{ia} [25]. From the argument relating eqs. (3.8) and (3.9), each π exchange and each ρ exchange is independently $SU(4)$ invariant, with corrections of $\mathcal{O}(1/N^2)$. One can think of the π and the ρ as being in different, incomplete $SU(4)$ multiplets, where the missing members of the multiplets would make negligible contributions to the nuclear force. (This argument only holds for S-wave interactions, for which $\{q_i \mathbf{G}_{ia}\}_1 \{q_j \mathbf{G}_{ja}\}_2 \sim q^2 \{\mathbf{G}_{ka}\}_1 \{\mathbf{G}_{ka}\}_2$.)

4. Conclusions

We conclude with a list of future directions for this line of investigation. One is to further the work of [1-3] relating the leading baryon interaction coefficients to the effective masses and properties of baryons in matter. In particular, one might analyze hypernuclei and Σ^- atoms in light of the predictions (1.6). One might also apply the predictions (1.6) to the controversy of how strangeness first appears in dense matter – in the form of kaon condensation, or in the form of hyperons [27]. However, these analyses require care: factors of 1/9 multiplying b in eq. (1.5) are formally of order $1/N^2$. Thus to leading order in N,

$c_5 = 2a$ and the other c_i's vanish giving rise to an accidental $SU(16)$ symmetry in the low energy self-interactions of the sixteen $J = 1/2$ baryon octet states. This $SU(16)$ symmetry is broken by subleading $SU(6)$ violating operators suppressed by $1/N$ or m_s, which are expected to be important phenomenologically [28]. Among the four nucleon isospin and spin states there is an accidental $SU(4)_{sm}$ symmetry which is much more robust, broken only at order $1/N^2$, and by isospin breaking.

Another suggestive line of inquiry is to extend the results of this Letter beyond the dimension six operators (1.2). In particular, one should understand how $SU(2f)$ breaking comes about in the $L \cdot S$ interactions, including analysis of the cases where $SU(4)_{sm}$ seems to work better than it should, in large nuclei where spin-orbit interactions are large — most notably the Franzini-Radicatti mass formula [29], [5]. Finally, it may prove fruitful to apply $SU(4)$ symmetry to the problem of the quenching of Gamow-Teller strengths in the shell model [17].

References

[1] S. Weinberg, Phys. Lett. 251B (1990) 288; Nucl. Phys. B363 (1991) 3; Phys. Lett. 295B (1992) 114

[2] C. Ordonez, U. van Kolck, Phys. Lett. 291B (1992) 459; C. Ordonez, L. Ray, U. van Kolck, Phys. Rev. Lett. 72 (1994) 1982; U. van Kolck, Phys. Rev. C49 (1994) 2932

[3] M. J. Savage, M. B. Wise, hep-ph/9507288

[4] E. Wigner, Phys. Rev. 51 (1937) 106, 947; *ibid.* 56 (1939) 519

[5] *Group Symmetries in Nuclear Structure*, J.C. Parikh, Plenum press, (1978)

[6] T.W. Donnelly and G.E. Walker, Ann. of Phys., 60 (1970) 209

[7] *Theoretical Nuclear Physics and Subnuclear Physics*, by J.D. Walecka, Oxford University Press (1995)

[8] J.P. Elliott, in *Isospin in Nuclear Physics*, page 73, ed. D.H. Wilkinson, North Holland Publishing, (1969)

[9] P. Vogel, M. R. Zirnbauer, Phys. Rev. Lett. 57 (1986) 3148

[10] P. Vogel, W.E. Ormand, Phys. Rev. C47 (1993) 623

[11] *Elements of Nuclear Physics*, W.E. Burcham, Longman Group Ltd, (1979)

[12] F. Ajzenberg-Selove, Nucl. Phys. A475 (1987) 1

[13] *The Practitioner's Shell Model*, G.F. Bertsch, North-Holland/American Elsevier (1972).

[14] R. Sorensen, *private communication*

[15] G. Bertsch, *private communication*

[16] E.G. Adelberger *et al*, Phys. Rev. C27 (1983) 2833

[17] G. F. Bertsch, H. Esbensen, Rep. Prog. Phys. 50 (1987) 607

[18] G. 't Hooft, Nucl. Phys. B72 (1974) 461; *ibid.* B75 (1974) 461

[19] E. Witten, Nucl. Phys. B160 (1979) 57

[20] J.L. Gervais, B. Sakita, Phys. Rev. Lett. 52 (1984) 87

[21] R. Dashen, A. V. Manohar, Phys. Lett. 315B (1993) 425; E. Jenkins, Phys. Lett. 315B (1993) 441; E. Jenkins, A. V. Manohar, Phys. Lett. 335B (1994) 452; R. Dashen, E. Jenkins, A. V. Manohar, Phys. Rev. D49 (1994) 4713; *ibid.* D51 (1995) 2489

[22] R. Dashen, E. Jenkins, A. V. Manohar, Phys. Rev. D51 (1995) 3697

[23] C. D. Carone, H. Georgi, S. Osofsky, Phys. Lett. 322B (1994) 227; C. D. Carone, H. Georgi, L. Kaplan, D. Morin, Phys. Rev. 50 (1994) 5793

[24] M. A. Luty, J. March-Russell, Nucl. Phys. B426 (1994) 71

[25] M. P. Mattis, M. Mukherjee, Phys. Rev. Lett. 61 (1988) 1344; M. P. Mattis, E. Braaten, Phys. Rev. D39 (1989) 2737; N. Dorey, J. Hughes, M. P. Mattis, Phys. Rev. Lett. 73 (1994) 1211; N. Dorey, M. P. Mattis, hep-ph/9412373

[26] G.E. Brown, R. Machleidt, Phys. Rev. C50 (1994) 1731

[27] See, for example, R. Knorren, M. Prakash, P.J. Ellis, nucl-th/9506016; J. Schaffner, I. N. Mishustin, nucl-th/9506011

[28] D.B. Kaplan, A.V. Manohar, in preparation.

[29] P. Franzini, L.A. Radicatti, Phys. Lett. 13 (1963) 322